

# NORMAL FAMILIES AND LINEAR DIFFERENTIAL EQUATIONS

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By Marty's Criterion (see Ahlfors [1]), normality of any family  $\mathcal{F}$  of meromorphic functions on some domain  $D$  is equivalent to local boundedness of the corresponding family  $\mathcal{F}^\#$  of *spherical derivatives*

$$f^\# = \frac{|f'|}{1 + |f|^2}.$$

Recently, J. Grahl and S. Nevo<sup>(1)</sup> proved a normality criterion involving the spherical derivative by utilising the so-called Zalcman Lemma (see L. Zalcman [7]). At first glance it looks very surprising since it is based on a *lower* bound for the spherical derivative.

**THEOREM (GRAHL & NEVO [3]).** *Suppose all functions of the family  $\mathcal{F}$  satisfy  $f^\#(z) \geq \epsilon$  for some fixed  $\epsilon > 0$ . Then  $\mathcal{F}$  is normal.*

The aim of this note is to give a completely different proof, which has the advantage to yield explicit upper bounds for  $f^\#$ . It is based on a property equivalent to  $f^\#(z) > 0$ , namely *local univalence* of the function  $f$ , which again is equivalent to the fact that the corresponding *Schwarzian derivative*

$$S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2$$

is holomorphic on  $D$ .

**THEOREM.** *Let  $f$  be meromorphic on the unit disc  $\mathbb{D}$  satisfying  $f^\#(z) \geq \epsilon > 0$ . Then  $f$  has the form*

$$(1) \quad f = \frac{w_1}{w_2},$$

where the functions  $w_1$  and  $w_2$  are holomorphic on  $\mathbb{D}$  and satisfy

$$(2) \quad |w_1(z)|^2 + |w_2(z)|^2 \leq \frac{1}{\epsilon}, \quad \left| \frac{w_1}{w'_1} \quad \frac{w_2}{w'_2} \right| = 1, \quad \text{and} \quad \left| \frac{w_1}{w''_1} \quad \frac{w_2}{w''_2} \right| = 0.$$

Moreover,

$$(3) \quad f^\#(z) \leq \frac{2/\epsilon}{(1 - |z|)^2} \quad \text{and} \quad |S_f(z)| \leq \frac{4/\epsilon}{(1 - |z|)^3}$$

hold on  $\mathbb{D}$ .

**PROOF.** Since  $f^\#$  is non-zero,  $f$  is locally univalent and its Schwarzian derivative is holomorphic on  $\mathbb{D}$ . It is well known that this implies the representation (1), where  $w_1$  and  $w_2$  form a fundamental set of the linear differential equation

$$(4) \quad w'' + \frac{1}{2} S_f(z) w = 0.$$

Then the third condition in (2) always holds (reflecting the fact that the coefficient of  $w'$  vanishes identically), hence the Wronskian of any two solutions is constant.

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<sup>1</sup>I learned about this in a talk given by J. Grahl at the second Bavarian-Québec Mathematical Meeting & Tag der Funktionentheorie, November 22-27, 2010, University of Würzburg.

To make some definite choice we normalise by the second condition in (2), which makes the pair  $(w_1, w_2)$  unique up to sign and from which

$$f' = \frac{-1}{w_2^2} \quad \text{and} \quad f^\# = \frac{1}{|w_1|^2 + |w_2|^2},$$

hence the first condition in (2) follows. To prove (3) we just remark that from  $\begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = 1$  and the Cauchy-Schwarz inequality follows

$$f^\# = \frac{1}{|w_1|^2 + |w_2|^2} \leq |w_1'|^2 + |w_2'|^2,$$

while  $\frac{1}{2}S_f = \begin{vmatrix} w_1' & w_2' \\ w_1'' & w_2'' \end{vmatrix}$  yields

$$\frac{1}{2}|S_f| \leq |w_1'| |w_2''| + |w_1''| |w_2'|.$$

The standard Cauchy estimate

$$|w| \leq 1/\sqrt{\epsilon} \quad \Rightarrow \quad |w'| \leq \frac{1/\sqrt{\epsilon}}{1-|z|} \quad \text{and} \quad |w''| \leq \frac{1/\sqrt{\epsilon}}{(1-|z|)^2}$$

then gives the estimate in both cases of (3).  $\square$

#### REMARKS AND QUESTIONS.

- For  $\epsilon > 0$  fixed, the family  $\mathcal{F}_\epsilon$  of all functions  $f$  satisfying  $f^\# \geq \epsilon$ , and also the family  $S_{\mathcal{F}_\epsilon}$  of corresponding Schwarzian derivatives is compact, and

$$\Phi_\epsilon(r) = \sup\{f^\#(z) : |z| \leq r, f \in \mathcal{F}_\epsilon\} \leq 2\epsilon^{-1}(1-r)^{-2} \quad (0 \leq r < 1)$$

holds. To obtain a lower bound for  $\Phi_\epsilon$  we consider  $f(z) = \left(\frac{1+z}{1-z}\right)^{i\lambda}$  (Hille's example [4] showing that Nehari's univalence criterion [5] is sharp). It has spherical derivative  $f^\#(z) = \frac{\lambda}{|1-z^2|} \frac{1}{|f(z)| + |f(z)|^{-1}}$  and Schwarzian derivative  $S_f(z) = 2(1+\lambda^2)(1-|z|^2)^{-2}$ , and satisfies  $f^\#(z) > f^\#(\pm i) = \lambda/\cosh \frac{\pi}{2}\lambda$  and  $f^\#(x) = \lambda(1-x^2)^{-1}$  ( $-1 < x < 1$ ), from which

$$\Phi_\epsilon(r) \geq (\log(1/\epsilon) + O(\log \log(1/\epsilon))) (1-r)^{-1} \quad (0 < \epsilon < \epsilon_0, 0 < r < 1)$$

follows ( $\epsilon_0 \approx 0.42$  is the maximum of  $\lambda/\cosh \frac{\pi}{2}\lambda$  in  $0 < \lambda < \infty$ ). The true value of  $\Phi_\epsilon(r)$  has to remain open; is it  $C(\epsilon)(1-r)^{-1}$ ? The problem to determine  $\sup_{\mathcal{F}_\epsilon} |S_f(z)|$  also remains open.

- Since  $|w_1|^2 + |w_2|^2$  is subharmonic, the spherical derivative satisfies the minimum principle: If  $f$  is meromorphic on some domain  $D$ , then  $f^\#$  has no local minima except at the critical points of  $f$  (see also [3], Prop. 4.) Actually,  $-\log f^\#$  is subharmonic off the zeros of  $f^\#$ .

- By Thm. 3 of [3],  $\mathcal{F}_\epsilon = \emptyset$  if  $\epsilon > 1/2$ , while  $\mathcal{F}_{1/2}$  consists of the rotations of the Riemann sphere. Using  $\begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = 1$  and  $1/f^\# = |w_1|^2 + |w_2|^2 \geq 2|w_1||w_2|$ , this may be shown as follows (similar to [3]): We first suppose  $w_1(0) = 0$  and set  $v(z) = w_2(z)w_1(z)/z$ . Then  $v(0) = w_1'(0)w_2(0) = -1$ , hence  $\max_{|z|=r} |v(z)| \geq 1$  holds by the maximum principle, this implying  $\min_{|z|=r} |z|f^\#(z) \leq 1/2$  and  $\inf_{\mathbb{D}} f^\#(z) \leq 1/2$ .

Also  $\inf_{\mathbb{D}} f^\#(z) = 1/2$  gives  $|v(z)| \leq 1 = |v(0)|$ , thus  $v(z) \equiv v(0) = -1$ . This, however, is only possible if  $w_1(z) = cz$  and  $w_2(z) = -1/c$ , hence  $f(z) = c^2 z$ , and from  $\inf_{\mathbb{D}} f^\#(z) = |c|^2/(1 + |c|^4)$  then follows  $|c| = 1$ . Without the normalisation  $f(0) = 0$ ,  $\inf_{\mathbb{D}} f^\# = 1/2$  implies that  $f$  is a rigid motion of the sphere.

• The upper bound for  $f^\#$  may be slightly improved. Given  $z \in \mathbb{D}$  we may assume  $f(z) = 0$  by rotating the Riemann sphere. We then have  $f^\#(z) = |w_2(z)|^{-2}$  and  $w_1'(z)w_2(z) = -1$ , hence  $f^\#(z) = |w_1'(z)|^2$ . By the Schwarz-Pick lemma (thanks to J. Grahl for the keyword) applied to  $\sqrt{\epsilon}w_1$  we thus obtain

$$f^\#(z) = |w_1'(z)|^2 \leq \frac{1/\epsilon}{(1 - |z|^2)^2}.$$

• The representation (1) together with the first condition in (2) implies that  $f$  has bounded *Nevanlinna characteristic* ( $f$  is called *of bounded type*), so that by the *Ahlfors-Shimizu formula*

$$\lim_{r \rightarrow 1} T(r, f) = \frac{1}{\pi} \int_0^1 (1 - \rho) \int_0^{2\pi} f^\#(\rho e^{i\theta})^2 d\theta d\rho$$

is finite (see Nevanlinna [6]). We note, however, that there are functions of bounded type having spherical derivative growing arbitrarily fast, see [2]. Thus, although normal functions [satisfying  $f^\#(z) = O((1 - |z|)^{-1})$  as  $|z| \rightarrow 1$ ] have Nevanlinna characteristic  $T(r, f) = O(-\log(1 - r))$ , there are functions of bounded type that are not normal.

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